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THE MODE OF ATTACHED COMPRESSION SHOCK AT EDGES OF A V-SHAPED WING

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The problem of supersonic flow past a V-shaped wing is considered in transonic approximation. The behavior of solution for modes close to the flow with a plane compression shock stretched over the [wing] edges is investigated in linear formulation. The mathematical problem reduces to the Riemann boundary value problem for analytic functions. The flow properties are investigated at small perturbations of the oncoming stream velocity. It is shown that when a plane compression shock at the wing edges in a plane normal to the edge isweak, the character of the flow is unaffected by variation of the Mach number. If, however, the shock in the main flow is strong, a rearrangement of the stream pattern is induced by a change of flow parameters. The considered problem is related to that of flow around star-shaped bodies investigated in [1 —5]; the processs of rearrangement of the mode of flow past a V-shaped wing was analyzed by numerical methods in [6] and investigated experimentally in [7].

Star-shaped bodies are bodies of least drag whose magnitude depends on the character of flow around them. A star-shaped body may be considered as composed of a number of V-shaped wings, for which it is possible to derive a class of exact solutions with a plane shock at their edges. For this we consider the plane supersonic flow past a wedge. We draw through the velocity vector behind the compression shock two planes symmetric about the normal to the wedge. Parts of these planes compresed between the wedge surface and the compression shock form a V-shaped wing at whose edges a plane compression shock is formed. When the flow past the wedge is supersonic a weak compression shock is always present [8]. However, the analog of the polar of the shock equation in a plane normal to the wing edge shows that, depending on the wing apex angle, the exact solution corresponds either to a weak or to a strong compression shock. A boundary value problem, dependent on the type of shock in the plane normal to the wing edge, is formulated for the determination of the flow pattern at small perturbations in the flow with a plane compression shock at the wing edges. The flow properties are determined by the conditions of solvability of that problem.

1. Let us consider this problem in the transonic approximation. A plane compression shock appear at the edges of a V-shaped wing in a slightly supersonic stream. In a coordinate system attached to the wing edge with the x-axis on the wing edge and the y-axis in the wing plane of symmetry (see Fig. 1), the components of transonic velocity $\mathbf{U} = (1 + u, v, w)$ (normalized with respect to the speed of sound) upstream and downstream of the shock wave are, respectively $u = u_{01}$, $v = v_{01}$, $w = 0$ and $u = u_{02}$, $v = 0$ and $w = 0$. We shall call this solution unperturbed.

The solution of the problem for small perturbations of the oncoming stream velocity is obtained by the method of the small parameter. If ε denotes the small parameter,

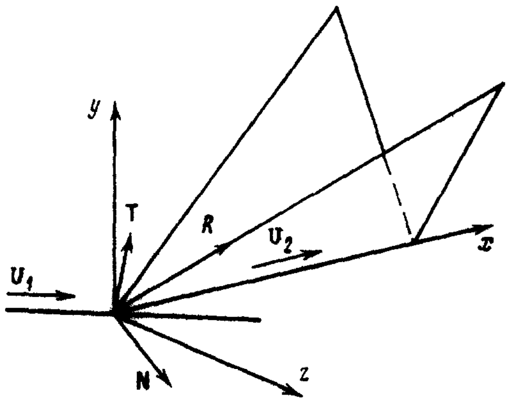

Fig. 1

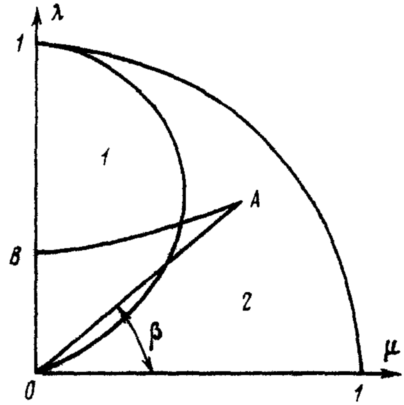

Fig. 2

the velocity components upstream and downstream of the shock wave become $u = u_{01}(1+\varepsilon)$, $v = v_{01}$, and $w = 0$ and $u = u_{02} + u'$, $v = v'$, and $w = w'$. In the transonic approximation the considered flow is potential, and its potential satisfies the equation

$$-\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Since the flow is conically symmetric and all solutions depend on variables $\xi = z/x$ and $\eta = y/x$, we introduce in the analysis the potential $\varphi(\xi,\eta) = \Phi(x, y, z)/x$. The velocity components are defined as follows:

$$u = \varphi - \xi\varphi_\xi - \eta\varphi_\eta = \varphi - \xi w - \eta v, \quad v = \varphi_\eta, \quad w = \varphi_\xi \tag{1.1}$$

We represent the potential φ in the form $\varphi = \varphi_0 + \varphi' = u_{02} + \varphi'$, where $\varphi_0 = u_{02}$ is the potential of the unperturbed flow. The linearized equations for φ' is of the form

$$(1-k^2\xi^2)\frac{\partial^2\varphi'}{\partial\xi^2} - 2k^2\xi\eta\frac{\partial^2\varphi'}{\partial\xi\partial\eta} + (1-k^2\eta^2)\frac{\partial^2\varphi'}{\partial\eta^2} = 0. \tag{1.2}$$

$(u_{02} = k^2)$. Its equivalent system is

$$(1-k^2\xi^2)\frac{\partial w'}{\partial\xi} - 2k^2\xi\eta\frac{\partial v'}{\partial\xi} + (1-k^2\eta^2)\frac{\partial v'}{\partial\eta} = 0, \quad \frac{\partial v'}{\partial\xi} = \frac{\partial w'}{\partial\eta} \tag{1.3}$$

From (1.1) and (1.2) we obtain for w' the equation

$$(1-k^2\xi^2)\frac{\partial^2 w'}{\partial\xi^2} - 2k^2\xi\eta\frac{\partial^2 w'}{\partial\xi\partial\eta} + (1-k^2\eta^2)\frac{\partial^2 w'}{\partial\eta^2} = 2k^2\left(\xi\frac{\partial w'}{\partial\xi} + \eta\frac{\partial w'}{\partial\eta}\right) \tag{1.4}$$

The mode of flow with a strong compression shock in the plane normal to the wing edge belongs to the ellipticity region ($\xi^2 + \eta^2 < 1/k^2$) of Eq. (1.4) (see Sect. 3). Let us formulate the boundary value problem for Eq. (1.4). Region T in which we seek the solution is represented by the triangle formed by traces of wing surface, the shock wave, and the plane of symmetry, and lying within a circle of radius $1/k$.

The equation of compression shock in the plane (ξ, η) in the unperturbed flow is of the form $\eta = \eta_0 = \text{const}$. The equation of shock for the perturbed flow can be represented as $\eta = \eta_0 + \eta'$ with

$$\eta' = 0 \tag{1.5}$$

at point A, which conforms to the assumption that for small perturbations the shock remains attached to the wing edge. At the compression shock we have the conditions

$$[u + \xi w + \eta v] = [\varphi] = 0 \tag{1.6}$$

$$[u]^2 u_* = [w]^2 + [v]^2 \tag{1.7}$$

the first of which defines the continuity of the potential and the second is of the form of the shock polar. In these equations $[f]$ is the jump of f at transition through the shock wave, and u_* is the half of the sum of values of u upstream and downstream of the shock. From (1.6) we obtain conditions of continuity of the unperturbed flow potential and of the potential of perturbations

$$u_{01} + \eta_0 v_{01} = u_{02}, \quad \varphi' = \varepsilon u_{01} + \eta' v_{01} = u' + \eta_0 \partial\varphi' / \partial\eta + \xi \partial\varphi' / \partial\xi \tag{1.8}$$

Condition (1.7) decomposes into the equation of the shock polar for the unperturbed flow and for perturbation velocity components

$$(u_{02} - u_{01})^2 (u_{02} + u_{01}) / 2 = v_{01}^2, \quad au' + v' + b = 0 \tag{1.9}$$

$$2v_{01}a = (u_{02} - u_{01})^2 / 2 + u_{02}^2 - u_{01}^2$$

$$2v_{01}b = \varepsilon u_{01} [(u_{02} - u_{01})^2 / 2 - u_{02}^2 + u_{01}^2]$$

From (1.8) and (1.9) we have

$$v_{01} = 2(c^2 - 1) / \eta_0^3, \quad u_{01} = (2 - c^2) / \eta_0^2, \quad u_{02}\eta_0^2 = c^2 \tag{1.10}$$

Using (1.1) we rewrite the last of relations (1.9) in the form $a\varphi' + \varphi_\eta' (1 - a\eta_0) - \varphi_\xi' a\xi + b = 0$. Differentiating this along the shock wave we obtain for Eq. (1.4) the following boundary condition:

$$\eta = \eta_0, \quad a\xi w_\xi' - (1 - a\eta_0) w_\eta' = 0 \tag{1.11}$$

At the wing surface the condition of impermeability $v' = w' \operatorname{tg} \beta$ applies. By differentiating it along the wing, and using Eq. (1.3) we obtain a boundary condition of the form

$$\eta - \xi \operatorname{tg} \beta = 0, \quad (2 - k^2\xi^2 / \cos^2 \beta) \operatorname{tg} \beta w_\xi' - (1 - \operatorname{tg}^2 \beta + k^2\xi^2 \operatorname{tg}^2 \beta / \cos^2 \beta) w_\eta' = 0 \tag{1.12}$$

Since at the plane of symmetry $w' = 0$, hence the condition

$$\xi = 0, \quad w_\eta' = 0 \tag{1.13}$$

Note that the formulation of the boundary value problem in the case of exact equations of gasdynamics is similar (except that pressure is substituted for the potential).

2. Let us reduce problem (1.4), (1.11) — (1.13) to the investigation of Riemann's problem on a circle. The coordinate transformation

$$\xi = 2\mu / [k(1 + \mu^2 + \lambda^2)], \quad \eta = 2\lambda / [k(1 + \mu^2 + \lambda^2)] \tag{2.1}$$

which maps the interior of the circle of radius $1 / k$ on to the interior of a unit circle in the plane (μ, λ), reduces Eq. (1.4) in the ellipticity region to the Laplace equation. Region T is then mapped onto region OAB (Fig. 2) and the straight line $\eta = \eta_0$ onto curve $\lambda = [1 - \sqrt{1 - c^2(1 + \mu^2)}] / c$. The boundary sections OA and OB are only subjected to stretching, since transformation (2.1) does not affect polar angles. Point A is the intersection of curves $\lambda = [1 - \sqrt{1 - c^2(1 + \mu^2)}]/ c$ and $\lambda = \mu \operatorname{tg} \beta$. Boundary conditions for the Laplace equations on OA, AB and BO are, respectively,

$$w_\mu' - w_\lambda' [\mu^2 + \cos^4 \beta (1 - \operatorname{tg}^2 \beta)] / [2 \operatorname{tg} \beta \cos^4 \beta] = 0 \tag{2.2}$$
$$w_\mu' \mu [(1 + c^2) c - 2\lambda] - w_\lambda' [\lambda (1 - c^2) c - 2\mu^2] = 0, \quad w_\lambda' = 0$$

Note that the boundary conditions are of the form $S w_\mu' - L w_\lambda' = 0$. Thus (2.2) is a problem with a directional derivative for the Laplace equation, which reduces to the Hilbert problem for analytic functions. We set $f = w_\mu'$, and $g = - w_\lambda'$. Function $F(z) = f(\mu, \lambda) + ig(\mu, \lambda)$ is analytic and satisfies at the region boundary the discontinuity boundary condition

$$S(\tau) f(\tau) + L(\tau) g(\tau) = 0 \tag{2.3}$$

Let function $r = R(z)$ conformally map region OAB onto a unit circle. Let us formulate the Hilbert problem for the circle. Boundary condition (2.3) in the r-plane is of the form

$$S_1(t) f_1(t) + L_1(t) g_1(t) = 0 \tag{2.4}$$
$$S_1(R(\tau)) = S(\tau), \quad L_1(R(\tau)) = L(\tau), \quad f_1(R(\tau)) = f(\tau)$$
$$g_1(R(\tau)) = g(\tau)$$

At points t_1, t_2, and t_3 which correspond to points O, A, and B functions $S_1(t)$ and $L_1(t)$ are discontinuous. Solution of the Hilbert problem for a circle coincides with the inner solution of the corresponding Riemann's problem [9]

$$F_1^+(t) = G_1(t) F_1^-(t), G_1(t) = [S_1(t) + iL_1(t)] / [S_1(t) - iL_1(t)] \tag{2.5}$$

The investigation of the problem of flow past a V-shaped wing thus reduces to an investigation of Riemann's problem with discontinuous coefficient on the circle.

Let $(-\theta_k)$ be the jump of the argument of function $S_1(t) + iL_1(t)$ at the discontinuity point t_k and be equal to the jump of the argument of function $S(\tau) + iL(\tau)$ at the corresponding discontinuity point τ_k. The jump of the argument of $G_1(t)$ at point t_k is equal $(-2\theta_k)$. Riemann's problem with discontinuous coefficient can be reduced to Riemann's problem with continuous coefficient [9]

$$F_2^+(t) = G_2(t) F_2^-(t), \quad G_2(t) = \prod_{k=1}^{3} (t - r_0)^{-\gamma_k} G_1(t) \tag{2.6}$$

The solution of problem (2.5) in terms of solution of problem (2.6) is of the form

$$F_1^+(r) = \prod_{k=1}^{3} (r - t_k)^{\gamma_k} F_2^+(r), \quad \operatorname{Re} \gamma_k = \frac{\theta_k}{\pi} - \varkappa_k$$

where $\varkappa_k = [\theta_k / \pi]$, if boundedness of solution is required at point t_k, and $\varkappa_k = [\theta_k / \pi + 1]$ when integrable infinity is admissible at point t_k [9]. An essential requirement in the considered problem is that of integrability of $F(z)$ in the z-plane, which is equivalent to the requirement for w' to be bounded. Let us determine the behavior of solution in the z-plane. Function $R(z)$ which maps region OAB onto the circle can be represented in the neighborhood of corner points in the form $r = t_k + C_k (z - \tau_k)^{\pi/\alpha_k} + \ldots$, where α_k is the angle between the tangents to the boundary of region OAB at the corner point τ_k. The solution of the problem in the neighborhood of these points is then of the form $F(z) = F_{\mathrm{I}}^{+}(R(z)) = C_k' (z - \tau_k)^{\pi\gamma_k/\alpha_k} F_2^{+}(R(z))$. Hence the solution at points τ_k under condition

$$\mathrm{Re}\,(\pi\gamma_k / \alpha_k) > -1 \tag{2.7}$$

has an integrable infinity. Using this property we determine the numbers $\varkappa_k$. We shall consider two cases: $0 \leqslant \theta_k < \pi$ and $-\pi \leqslant \theta_k < 0$, and assume that for $0 \leqslant \theta_k < \pi$ $F_1^{+}(r)$ has an integrable infinity at point t_k. Then $\varkappa_k = 1$, and $\mathrm{Re}\,\gamma_k = \theta_k / \pi - 1$. It follows from (2.7) that integrability in the z-plane depends on the fulfilment of condition $\theta_k > \pi - \alpha_k$. When $\theta_k \leqslant \pi - \alpha_k$ the boundedness of the solution in r implies that $\varkappa_k = 0$, and then $\mathrm{Re}\,\gamma = \theta_k / \pi$ and condition (2.7) are satisfied. We pass to the case when $\pi \leqslant \theta_k < 0$. Assuming that $F_1^{+}(r)$ has an integrable infinity at point t_k we obtain $\varkappa_k = 0$ and $\mathrm{Re}\,\gamma_k = \theta_k / \pi$. It follows from (2.7) that integrability in the z-plane depends on the fulfilment of condition $\theta_k > -\alpha_k$. The boundedness of solution in r for $\theta_k \leqslant -\alpha_k$ implies that $\varkappa_k = -1$, hence $\mathrm{Re}\,\gamma_k = \theta_k / \pi + 1$ and condition (2.7) is satisfied.

Let us pass to the analysis of boundary conditions. Note that nowhere along the boundary of region OAB S and L vanish simultaneously. Since at the boundary of OAB $S \geqslant 0$, hence when moving around the contour OAB the argument of function $S + iL$ cannot vary by more than π on any section of that contour. Since the argument of function $S + iL$ is determined on every section of the boundary to within $2n\pi$ and the selection of the argument branch does not affect the final result, we assume the argument to be $\mathrm{arctg}\,(L / S)$. Then $|\theta_k| < \pi$ and the numbers $\varkappa_k$ are determined by the method indicated above. Formula (2.2) shows that by such selection of the argument branches the boundary condition discontinuity at point B is avoided. The problem is now reduced to the determination of numbers $\varkappa_k$ depending on the position of point A. We denote the quantities θ, α_k, and $\varkappa_k$ at points O and A by $\theta_1, \alpha_1, \varkappa_1$ and $\theta_2, \alpha_2, \varkappa_2$, respectively. We fix c and observe the changes of θ_k and α_k at points O and A when point A moves along the curve $\lambda = [1 - \sqrt{1 - c^2(1 + \mu^2)}] / c$. Using formulas (2.2) and the method of determining numbers $\varkappa_k$ we find that independently of the position of point A the condition $\alpha_1 < \theta_1 < \pi - \alpha_1$ is satisfied, hence it is necessary to set $\varkappa_1 = 0$ at point O in order to ensure the integrability of solution in the z-plane. If point A lies in the region to the left of the curve that corresponds to the condition $\sin^2\beta = (c^2 + 1) / 2$, we have $-\pi < \theta_2 < -\alpha_2$, and if it is to the right of that curve we have $-\alpha_2 < \theta_2 < \pi - \alpha_2$. We call these regions strong and weak, respectively (regions 1 and 2 in Fig. 2). The method of determination of $\varkappa_k$ implies that to ensure the integrability of solution in the z-plane it is necessary to set $\varkappa_2 = -1$ in the strong region and $\varkappa_2 = 0$ in the weak one.

We denote the index of problem (2. 5) by $\varkappa$. It is known that

$$\varkappa = \sum_k \varkappa_k$$

i.e. in the strong region $\varkappa = -1$ and in the weak $\varkappa = 0$. In conformity with the theorem in [9] this imples that the problem is insoluble in the strong region, while in the weak region it has a single linearly independent solution. A trivial solution is to be disregarded, since it does not satisfy condition (1. 5).

3. We shall now clarify the physical meaning of derived solutions. In a plane normal to the wing edge the weak region corresponds to a weak compression shock, and the strong region to a strong shock to the analog of the shock polar. Using formulas (1. 10) and denoting the transonic velocity components by an asterisk, we obtain

$$U_{1R}^* = \sin\beta / \eta_0, \quad U_{1N}^* = (2 - c^2 - {}^1/_2 \sin^2\beta) / \eta_0^2, \quad U_{1T}^* = 0$$

$$U_{2R}^* = \sin\beta/\eta_0, \quad U_{2N}^* = (c^2 - {}^1/_2 \sin^2\beta)/\eta_0^2, \quad U_{2T}^* = 2(1-c^2)\cos\beta/\eta_0^3$$

where U_{iR}, U_{iN} and U_{iT} $(i=1,2)$ are velocity components in directions of $\mathbf{R}$, $\mathbf{N}$, and $\mathbf{T}$, where $\mathbf{R}$ is the unit vector along the wing edge, $\mathbf{N}$ is the unit vector whose direction coincides with that of vector $\mathbf{U}_1 - U_{1R}\mathbf{R}$ which lies in the plane normal to $\mathbf{R}$, and $\mathbf{T} = \mathbf{R} \times \mathbf{N}$ (see Fig. 1).

The transonic components satisfy the analog of the equation of the shock polar

$$U_{2T}^* = \sqrt{(U_{1N}^* + U_{2N}^* - U_{1R}^{*2}) / 2} \; (U_{1N}^* - U_{2N}^*) \tag{3.1}$$

When condition $dU_{2T}^* / dU_{2N}^* > 0$ is satisfied we have a strong compression shock and when $dU_{2T}^* / dU_{2N}^* < 0$ holds the shock is weak. From (3. 1) we have

$$\frac{dU_{2T}^*}{dU_{2N}^*} = \frac{-U_{1N}^* - 3U_{2N}^* + U_{1R}^{*2}}{\sqrt{8(U_{1N}^* + U_{2N}^* - U_{1R}^{*2})}}$$

Substituting into this equation the expressions for U_{1N}^*, U_{2N}^*, and U_{1R}^* we find that when condition $\sin^2\beta > (1 + c^2) / 2$ is satisfied the compression shock in a plane normal to the wing edge is strong, while under condition $\sin^2\beta < (1 + c^2) / 2$ is weak.

The solvability of the boundary value problem formulated here thus depends on the type of compression shock in a plane normal to the wing edge. If the shock is weak the flow pattern remains unchanged, i. e. the compression shock is distorted when perturbations are small, while if the shock is strong, the problem has no solution, which indicates a basic rearrangement of the flow pattern. The problem of the pattern of flow in the latter case requires a separate investigation.

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